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Abstract. Let E be a *UMD* Banach space, and L a positive self-adjoint operator in L^2 of Laplace type, whose imaginary powers L^{-it} form a C_0 -group of exponential growth $0 \leq \alpha < \pi$ on $L^p(E)$ where $1 < p < \infty$. Suppose $G(z)$ is holomorphic inside and on the boundary of the sector $\{z : |\arg z| \leq \phi\}$, and $z^\kappa G(z) \rightarrow 0$ uniformly as $z \rightarrow \infty$ for some $\kappa > 0$ and $\phi > \alpha$. Then $G(tL)$ ($t > 0$) defines a bounded family of linear operators on $L^p(E)$; and the maximal operator $f \mapsto \sup_{t>0} \|G(tL)f\|_E$ is bounded on the domain of $\log L$. The proof uses transference methods. These hypotheses hold for the maximal solution operators for the heat, wave and Schrödinger equations, and to Cesàro sums.

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1. Introduction and main results

Let X be a smooth manifold, and μ a σ -finite positive Radon measure on X . Suppose that L is an operator of Laplace type over X , so that L defines a positive and self-adjoint operator in a dense linear subspace of $L^2(\mu)$ with core $C_c^\infty(X)$; to avoid possible ambiguity, we further suppose that the spectral projection corresponding to the singleton set $\{0\}$ is zero. Any bounded and Borel measurable function $G : (0, \infty) \rightarrow \mathbb{C}$ gives rise to a bounded linear operator $G(L)$ on $L^2(\mu)$ by the spectral theorem.

The solutions to the abstract Cauchy problem $\partial u / \partial t = iLu$, $u(0, x) = f(x)$, for the Schrödinger equation are given by $u(t, x) = e^{itL}f(x)$; although this problem is well-posed for initial data $f \in L^2(\mu)$, one typically obtains μ -almost sure convergence to the initial data, $u(t, x) \rightarrow f(x)$ as $t \rightarrow 0$, only for f in a proper subspace of $L^2(\mu)$ [12, 16]. The papers [9, 5] introduced a technique for proving such results, based upon analysis of the maximal operators $f \mapsto \sup_{t>0} |G(tL)f(x)|$. Using the Mellin transform of G , one can reduce the analysis of the maximal operator to the group L^{iv} of imaginary powers of L . Typically this defines a C_0 -group on $L^p(\mu)$ ($1 < p < \infty$), which is not uniformly bounded for $p \neq 2$ [17]. In this paper we achieve new maximal theorems for operator families defined using such groups. To obtain estimates on various operators by transference methods, it

is necessary to restrict attention to a special class of Banach spaces which includes $L^p(\mu)$ ($1 < p < \infty$).

Definition. For E a complex Banach space, the Bochner-Lebesgue space $L^p(\mu; E)$ consists of strongly measurable E -valued functions f on X for which the norm $\|f\|_{L^p(\mu; E)} = \left(\int_X \|f(s)\|_E^p \mu(ds) \right)^{1/p}$ is finite. And E is said to be a *UMD* space, if the Hilbert transform

$$H(f)(t) = \text{PV} \int_{-\infty}^{\infty} f(t-s) \frac{ds}{\pi s} \quad (f \in L^p(\mathbb{R}; E)) \quad (1.1)$$

defines a bounded linear operator on $L^p(\mathbb{R}; E)$ for $1 < p < \infty$. See [3, 6, 7] for examples.

When E is a *UMD* Banach space, one seeks to extend the definition of L^{iv} to $L^{iv} \otimes Id_E$ on $L^p(\mu; E)$ by using the functional calculus of spectral integration [2, 3]; for notational simplicity, one then writes L^{iv} for this operator. The domain of the generator $i \log L$ of the group L^{iu} in $L^p(\mu; E)$ has the usual graph norm

$$\|f\|_{\mathbb{D}(\log L)} = \|f\|_{L^p(\mu; E)} + \|(\log L)f\|_{L^p(\mu; E)}.$$

Using a much improved version of the technique of [5], we obtain in section two results for maximal functions associated with operator families $G(tL)$ on $L^p(\mu; E)$. Our first application Theorem 3.1 relates to the maximal operator for the holomorphic extension of the heat semigroup $H_t = e^{-tL}$, when L^{iv} is of exponential growth of rate less than $\pi/2$. In section three we also obtain, under stronger hypotheses, bounds on the solution operator for the wave and Schrödinger equations. Our main result is:

Theorem 1.1. Suppose that L^{iv} ($v \in \mathbb{R}$) defines a C_0 -group on $L^p(\mu; E)$ for some $1 < p < \infty$ and *UMD* Banach space E , with

$$\|L^{iv}\|_{B(L^p(\mu; E))} \leq C(1 + |v|^\beta) \quad (v \in \mathbb{R}) \quad (1.2)$$

where $\beta < 1/2$. Then (a) the wave sine family $\sin t\sqrt{L}/t\sqrt{L}$ ($t > 0$) defines a uniformly bounded family of linear operators on $L^p(\mu; E)$ and the sublinear maximal operator $W_* : \mathbb{D}_{L^p(\mu; E)}(\log L) \rightarrow L^p(\mu)$ is bounded, where

$$W_* : f \mapsto \sup_{t>0} \left\| \frac{\sin t\sqrt{L}}{t\sqrt{L}} f(x) \right\|_E \quad (f \in \mathbb{D}_{L^p(\mu; E)}(\log L)). \quad (1.3)$$

(b) Likewise, the family of Schrödinger sine operators $\sin tL/tL$ ($t > 0$) is uniformly bounded on $L^p(\mu; E)$ and the sublinear maximal operator $S_* : \mathbb{D}_{L^p(\mu; E)}(\log L) \rightarrow L^p(\mu)$ is bounded, where

$$S_* : f \mapsto \sup_{t>0} \left\| \frac{\sin tL}{tL} f(x) \right\|_E \quad (f \in \mathbb{D}_{L^p(\mu; E)}(\log L)). \quad (1.4)$$

In section four we derive some consequences for the Euclidean Laplace operator.

We shall show that the maximal operator W_* of (1.3) has similar properties to the Hardy-Littlewood maximal operator. The relations between the various maximal operators are discussed in section six.

Our main Maximal Lemma 2.3 also has surprising consequences for Cesàro summability of orthogonal series. Let (φ_n) ($n \in \mathbb{N}$) be an orthonormal basis of $L^2(\mu)$ for some probability measure on X , and for $f \in L^2(\mu)$ write $f \sim \sum_n a_n \varphi_n$ for the orthogonal series generated by f . For a real sequence of $\lambda_n \geq 1$ with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, we can introduce a self-adjoint and positive operator in $L^2(\mu)$ by setting $L\varphi_n = \lambda_n \varphi_n$ and extending linearly. In section five we prove results including the following:

Theorem 1.2. *Suppose $\delta > 0$ and that f belongs to the domain of $\log L$, so $\sum_n (1 + \log \lambda_n)^2 |a_n|^2 < \infty$. Then $\sum_n a_n \varphi_n(x)$ is almost everywhere (C, δ) summable to $f(x)$, so*

$$\sum_{n: \lambda_n \leq N} \left(1 - \frac{\lambda_n}{N}\right)^\delta a_n \varphi_n(x) \rightarrow f(x) \quad (1.5)$$

μ -almost everywhere as $N \rightarrow \infty$.

In this context, it is natural to have maximal operators defined only on the domain of $\log L$, as illustrated by the theorem of Menshov [15]:

Given $\varepsilon > 0$, there exists an orthonormal basis (φ_n) ($n \in \mathbb{N}$) of $L^2[0, 1]$ for which the maximal partial sum operator $\sum_n a_n \varphi_n \mapsto \sup_{N(x)} |\sum_{n=1}^{N(x)} a_n \varphi_n(x)|$ is unbounded $\mathbb{D}_{L^2}((\log L)^{1-\varepsilon}) \rightarrow L^1[0, 1]$ where L is determined by $\lambda_n = n + 1$ ($n \in \mathbb{N}$).

The proof of the main Maximal Lemma in section two involves a functional calculus argument which the reader may find more congenial when expressed in the language of Fourier transforms. For $\kappa, \beta > 0$, let $\mathbb{A}_{\beta, \kappa}$ be the space of holomorphic functions $g(z)$ on $\Omega_\beta = \{z : |\Im z| < \beta\}$ which have (a) $e^{\kappa z} g(z)$ and (b) $e^{-\kappa z} g'(z)$ bounded on Ω_β . It is easy to verify that any such g is itself bounded on Ω_β , and hence that $\mathbb{A}_{\beta, \kappa}$ forms a Banach algebra under pointwise multiplication for the norm

$$\|g\|_{\mathbb{A}_{\beta, \kappa}} = \sup_{z \in \Omega_\beta} \{|g(z)|\} + \sup_{z \in \Omega_\beta} \{|e^{\kappa z} g(z)|\} + \sup_{z \in \Omega_\beta} \{|e^{-\kappa z} g'(z)|\}. \quad (1.6)$$

Theorem 1.3. *Let E be a UMD Banach space, and $T_t = e^{itA}$ a C_0 -group of operators on E , with $\|T_t\|_{B(E)} \leq Me^{\alpha|t|}$ ($t \in \mathbb{R}$). Then, for any $\beta > \alpha$ and $\kappa > 0$, there is a bounded homomorphism $\mathbb{A}_{\beta, \kappa} \rightarrow B(E) : g \mapsto g(A)$ for any $\kappa > 0$.*

The main idea in the proof is that $\hat{g}(u + iv) = \int_{-\infty}^{\infty} g(x + iy) e^{-i(x+iy)(u+iv)} dx$ is independent of $-\beta < y < \beta$ by Cauchy's Theorem, and defines a holomorphic function on

$\{u + iv : 0 < v < \kappa, -\infty < u < \infty\}$. Using (b) and an integration by parts argument, one can extend \hat{g} to a meromorphic function on Ω_κ whose only possible singularity is a simple pole at zero. Further, for any $\beta' < \beta$ and $\kappa' < \kappa$, the extended function $\hat{g}(u + iv)$ decays at exponential rate β' uniformly on $\Omega_{\kappa'}$ as $u \rightarrow \pm\infty$.

The functional calculus map is

$$g \mapsto \lim_{\varepsilon \rightarrow 0+} \frac{1}{2\pi} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \hat{g}(u) T_u du - \frac{i}{2} \text{Res}(\hat{g}; 0) Id_E, \quad (1.7)$$

wherein the integrals converge at infinity on account of the exponential decay of the Fourier transform, while the singularity at zero may be treated by transference methods.

2. Maximal Lemma for imaginary powers

Throughout this section, we suppose $1 < p < \infty$, and E is a *UMD* Banach space. We also suppose that the group of imaginary powers L^{iv} ($v \in \mathbb{R}$) defines a C_0 -group on $L^p(\mu; E)$ which satisfies

$$\|L^{iv}\|_{B(L^p(\mu; E))} \leq M_L e^{\alpha|v|} \quad (v \in \mathbb{R}). \quad (2.1)$$

Stein [17, p. 58] shows that the negative generator of any symmetric diffusion semigroup e^{-tL} satisfies (2.1) with $\alpha \leq \pi/2$ when $E = \mathbb{C}$; this gives us our main source of examples. See also [4, 5.2].

Using the Mellin transform, we shall show that $G(L) \otimes Id_E$ may be extended from $C_c^\infty(X) \otimes E$ to define a bounded linear operator $G(L)$ on $L^p(\mu; E)$, provided G satisfies various technical conditions. We first impose a strong hypothesis, to be weakened later:

(i) $G(x)/x^{1+\gamma}$ is integrable over x in $(0, \infty)$ for $-\delta < \gamma < \delta$, for some $\delta > 0$.

Then the Mellin transform may be defined by $\tilde{G}(s) = \int_0^\infty x^{s-1} G(x) dx$, where the integral is absolutely convergent, and is a holomorphic function of s on the strip $\{s : -\delta < \Re s < \delta\}$. The Mellin inversion formula [20, p. 46] gives

$$G(\lambda) = \lim_{R \rightarrow \infty} \int_{\gamma-iR}^{\gamma+iR} \tilde{G}(s) \lambda^{-s} ds / 2\pi i \quad (-\delta < \gamma = \Re s < \delta) \quad (2.2)$$

at all points of differentiability of G . Similarly, if $\tilde{G}(\gamma + iv)$ is integrable over $v \in (-\infty, \infty)$ and differentiable at $\gamma + it$, then $G(x) = \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{G}(s) x^{-s} ds / 2\pi i$ satisfies $\tilde{G}(\gamma + it) = \lim_{\eta \rightarrow 0+, R \rightarrow \infty} \int_\eta^R x^{\gamma+it-1} G(x) dx$.

For $0 < \phi < \pi$ we introduce the sector $K_\phi = \{z : z \neq 0, |\arg z| \leq \phi\}$ and consider the complex functions G which are continuous on K_ϕ and holomorphic on its interior. These assumptions will be in force throughout this section.

We recall the proof of a fundamental result [5, 9], in which we further suppose:

- (ii) $G(z) \rightarrow 0$ as $|z| \rightarrow 0$ and as $|z| \rightarrow \infty$ in K_ϕ , uniformly in $|\arg z| \leq \phi$;
- (iii) $r^{-1}G(re^{\pm i\phi})$ are integrable over $r \in (0, \infty)$.

Proposition 2.1. *Suppose that G satisfies (i), (ii) and (iii), for some $\pi > \phi > \alpha$. Then $G(tL)$ ($t > 0$) defines a uniformly bounded family of operators on $L^p(\mu; E)$, and the maximal operator $f \mapsto \sup_{t>0} \|G(tL)f(x)\|_E$ defines a bounded sublinear operator $L^p(\mu; E) \rightarrow L^p(\mu)$.*

Proof. Using Cauchy's theorem and condition (ii), we can turn the line of integration in the definition of the Mellin transform to $\arg z = \operatorname{sgn}(v)\phi = \psi$ and thereby achieve

$$\tilde{G}(iv) = e^{-|v|\phi+i\psi} \int_0^\infty r^{iv-1} G(re^{i\psi}) dr \quad (v \in \mathbb{R}). \quad (2.3)$$

By the inversion formula, we can then write

$$G(tL)f = \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{G}(iv) t^{-iv} L^{-iv} f dv \quad (f \in L^p(\mu; E), t > 0). \quad (2.4)$$

For any $x \in X$, we have by the triangle inequality

$$\sup_{t>0} \|G(tL)f(x)\|_E \leq \frac{1}{2\pi} \int_{-\infty}^\infty |\tilde{G}(iv)| \|L^{-iv} f(x)\|_E dv, \quad (2.5)$$

since $|t^{-iv}| = 1$. Taking the $L^p(\mu)$ norm of both sides gives

$$\begin{aligned} \left\| \sup_{t>0} \|G(tL)f\|_E \right\|_{L^p(\mu)} &\leq \frac{1}{2\pi} \int_{-\infty}^\infty |\tilde{G}(iv)| \|L^{-iv} f\|_{L^p(\mu; E)} dv \\ &\leq \frac{1}{2\pi} \int_{-\infty}^\infty |\tilde{G}(iv)| \|L^{-iv}\|_{B(L^p(\mu; E))} dv \times \|f\|_{L^p(\mu; E)} \end{aligned} \quad (2.6)$$

for any $f \in L^p(\mu; E)$; and the latest integral is convergent since the exponential decay rate of the Mellin transform exceeds the growth rate of the group by (2.3) and (iii).

Unfortunately, neither condition (i) nor (ii) is satisfied in many cases of interest, such as Theorems 1.1, 3.1 and Corollary 4.1(a) and (b), wherein the Mellin transform $\tilde{G}(s)$ has a simple pole at $s = 0$. Consequently, the constant $C(p, \alpha)$ of Cowling's fundamental inequality [9, p. 276] becomes infinite, and one cannot control (the maximal function of) the family $G(tL)$ ($t > 0$) by a simple application of the triangle inequality. The main technical contribution of the present paper is to use transference methods to overcome this difficulty. This involves restricting the range of values of p , restricting the class of Banach spaces and defining the maximal operator on a proper subspace of $L^p(\mu; E)$.

Lemma 2.2. Suppose that for some $\kappa > 0$ and $0 < \phi < \pi$:

(iv) $G(z)$ is holomorphic with continuous derivative inside and on the boundary of K_ϕ , including $z = 0$; and

(v) $G(re^{i\theta})r^\kappa \rightarrow 0$ as $r \rightarrow \infty$, uniformly for $|\theta| \leq \phi$.

Then $\tilde{G}(s)$ extends to define a holomorphic function on the strip $\{s : -1 < \Re s < \kappa\}$, apart possibly from a simple pole at $s = 0$. Further, $G(s)$ decays to zero at exponential rate ϕ , uniformly for $-1/2 \leq \Re s \leq \kappa/2$ as $\Im s \rightarrow \pm\infty$.

Proof. For $0 < \Re s < \kappa$, the Mellin transform integral is absolutely convergent and defines a holomorphic function on this strip by Morera's theorem. By applying Cauchy's Theorem to $z^{s-1}G(z)$ on the contour made up of the straight line segments $[\delta, R]$, $[\delta e^{i\theta}, R e^{i\theta}]$ and the circular arcs joining δ to $\delta e^{i\theta}$ and R to $R e^{i\theta}$ respectively, we can turn the line of integration to $\arg z = \theta$ for any $-\phi \leq \theta \leq \phi$. Letting $\delta \rightarrow 0+$ and $R \rightarrow \infty$, we derive the identity

$$\tilde{G}(s) = e^{i\theta s} \left(\frac{G(e^{i\theta})}{s} - \frac{1}{s} \int_0^1 r^s G'(re^{i\theta}) dr + \int_1^\infty r^{s-1} G(re^{i\theta}) dr \right) \quad (|\theta| \leq \phi, 0 < \Re s < \kappa) \quad (2.7)$$

by splitting the range of integration and integrating by parts.

One can use the right-hand side of (2.7) to continue $\tilde{G}(s)$ across the imaginary axis to a meromorphic function on $\{s : -1 < \Re s < \kappa\}$, whose only possible singularity is a simple pole at $s = 0$. On account of the initial exponential factor, it also follows that $|\tilde{G}(u + iv)| \leq C_G e^{-\phi|v|}$ for $-1/2 \leq u \leq \kappa/2$ and $|v| \geq 1$ for some $C_G < \infty$.

Maximal Lemma 2.3. Let G be as in Lemma 2.2, and suppose L^{iu} satisfies (2.1), for some $\pi > \phi > \alpha$. Then $G(tL)$ ($t > 0$) defines a uniformly bounded family of linear operators on $L^p(\mu; E)$; further, the maximal operator $f \mapsto \sup_{t>0} \|G(tL)f\|_E$ defines a bounded sublinear operator $\mathbb{D}_{L^p(\mu; E)}(\log L) \rightarrow L^p(\mu)$.

Proof. For $\eta > 0$ sufficiently small, the Mellin inversion formula may be applied to $\tilde{G}(s + \eta)$, the Mellin transform of $x^\eta G(x)$. One can then use Cauchy's theorem to deform the line of integration into a curve avoiding $s = 0$. On letting $\eta \rightarrow 0$, this contour may in turn be replaced by the sum of a principal value integral and the contribution of the pole at zero. Consequently, for $f \in L^p(\mu; E)$ and $t > 0$ we can use the Mellin inversion formula to write

$$\begin{aligned} G(tL)f &= \frac{1}{2\pi} \int_1^\infty + \int_{-\infty}^1 \tilde{G}(iv)t^{-iv} L^{-iv} f dv \\ &\quad + \text{PV} \frac{1}{2\pi} \int_{-1}^1 \tilde{G}(iv)t^{-iv} L^{-iv} f dv + 2^{-1} \text{Res}(\tilde{G}; 0)f. \end{aligned} \quad (2.8)$$

The integrals over $(1, \infty)$ and $(-\infty, -1)$ may be bounded by the argument of Proposition 2.1, whereas the final integral in (2.8) requires a more subtle argument: starting with the decomposition $\tilde{G}(iv) = b/iv + h(v)$, where b is a constant and $h(v)$ is continuous near to the simple pole at $v = 0$, we have an identity of principal value integrals

$$\int_{-1}^1 \tilde{G}(iv) t^{-iv} L^{-iv} f dv = \int_{-1}^1 h(v) t^{-iv} L^{-iv} f dv + b \int_{-1}^1 (tL)^{-iv} f \frac{dv}{iv}. \quad (2.9)$$

The first of these integrals is clearly bounded with bound independent of t as in Proposition 2.1. The last is the transferred finite Hilbert transform for the representation $v \mapsto (tL)^{-iv}$ of \mathbb{R} on $L^p(\mu; E)$, and by [3; 4, Theorem 4.1] this operator is bounded with bound independent of t . Together these estimates show that $G(tL)$ ($t > 0$) defines a uniformly bounded family of linear operators.

To control the associated maximal operator, we take $f \in \mathbb{D}_{L^p(\mu; E)}(\log L)$ and write the last integral in (2.8) as a sum of principal value integrals

$$\begin{aligned} \int_{-1}^1 \tilde{G}(iv) t^{-iv} L^{-iv} f dv &= \int_{-1}^1 h(v) t^{-iv} L^{-iv} f dv + \int_{-1}^1 \frac{t^{-iv} - 1}{iv} (L^{-iv} - I) f dv \\ &\quad + \int_{-1}^1 \frac{t^{-iv}}{iv} f dv + \int_{-1}^1 L^{-iv} f \frac{dv}{iv} - \int_{-1}^1 \frac{dv}{iv} f. \end{aligned} \quad (2.10)$$

The first of these integrals may be treated as in Proposition 2.1. The latest integral is zero; the second last is independent of $t > 0$ and is the transferred finite Hilbert transform, hence may be controlled as in (2.9); the third last involves $\int_0^1 \sin(v \log t) dv/v$, which is bounded independent of $t > 0$. It remains to control the second integral, which may be written as

$$\int_{-1}^1 (t^{-iv} - 1) \frac{1}{iv} \int_0^v L^{-iw} dw (-i \log L) f dv \quad (2.11)$$

by definition of the group generator, wherein $|t^{-iv} - 1| \leq 2$ and

$$\left\| \frac{1}{iv} \int_0^v L^{-iw} dw \right\|_{B(L^p(\mu; E))} \leq \frac{M_L}{|v|} \int_0^{|v|} e^{w\alpha} dw \leq M_L e^\alpha \quad (|v| \leq 1) \quad (2.12)$$

by (2.1). Consequently we can take the supremum over $t > 0$, and then the $L^p(\mu)$ norm, to obtain

$$\left\| \sup_{t>0} \left\| \int_{-1}^1 \frac{t^{-iv} - 1}{iv} (L^{-iv} - I) f dv \right\|_E \right\|_{L^p(\mu)} \leq 2M_L e^\alpha \|(\log L) f\|_{L^p(\mu; E)}. \quad (2.13)$$

Combining all of these estimates gives the required bound on the maximal function.

3. Heat, wave and Schrödinger maximal theorems

Our first application of the Maximal Lemma is to the holomorphic semigroup e^{-wL} . This is associated with the heat equation, in the sense that $U(w, x) = e^{-wL}f(x)$ satisfies $\partial U/\partial w = -LU$ for w in a sector surrounding the semi-axis $\Re w > 0$.

Theorem 3.1. *Let L^{iv} ($v \in \mathbb{R}$) be a C_0 -group on $L^p(\mu; E)$ for some UMD Banach space E and $1 < p < \infty$, which satisfies (2.1) with $\alpha < \pi/2$; and let $0 < \psi < \pi/2 - \alpha$. Then $(-L)$ generates a holomorphic semigroup on $L^p(\mu; E)$, bounded in the sector $\{w : |\arg w| \leq \psi\}$; and there exists $C = C(E, p, \psi, \alpha) < \infty$ such that the associated maximal operator satisfies*

$$\left\| \sup_{w: |\arg w| \leq \psi} \|e^{-wL}f\|_E \right\|_{L^p(\mu)} \leq CM_L \left(\|f\|_{L^p(\mu; E)} + \|(\log L)f\|_{L^p(\mu; E)} \right) \quad (3.1)$$

for all f in the domain of the group's generator $i \log L$.

Proof. The existence of such a semigroup was established in [4, Corollary 4.2]. Let ω have $|\omega| < \psi < \pi/2 - \alpha$, and set $G_\omega(z) = \exp(-e^{i\omega}z)$. One can verify (v) with any $\phi < \pi/2 - \psi$, or calculate the Mellin transform explicitly. Indeed $\tilde{G}_\omega(s)$ equals $\Gamma(s)e^{-is\omega}$, which satisfies the asymptotic relation

$$|\tilde{G}_\omega(iv)| \asymp \left(\frac{2\pi}{|v|} \right)^{1/2} e^{\omega v} e^{-\pi|v|/2} \quad (v \rightarrow \pm\infty) \quad (3.2)$$

by Stirling's formula [19, 4.4.1]; hence $\tilde{G}_\omega(iv)$ is of exponential decay of rate at least $\pi/2 - \psi > \alpha$ as $v \rightarrow \pm\infty$. Further, $\tilde{G}_\omega(s)$ is holomorphic on a strip surrounding the imaginary axis, apart from a simple pole at zero with residue one.

By Lemma 2.3, $\exp(-e^{i\omega}tL)$ ($t > 0, |\omega| \leq \psi$) defines a uniformly bounded family of linear operators on $L^p(\mu; E)$, and the associated maximal operator is bounded from the domain of $\log L$ to $L^p(\mu)$.

3.2 Proof of Theorem 1.1 (a) The function $G(x) = \sin \sqrt{x}/\sqrt{x}$ ($x > 0$) has Mellin transform $\tilde{G}(s) = -2\Gamma(2s-1)\cos \pi s$, with poles at $s = -n$, with $n \geq 0$ an integer [20, p. 204]. Further, by Stirling's formula [19, 4.41], we have the asymptotic relation

$$\tilde{G}(iv) \asymp \frac{-\sqrt{\pi}}{|v|^{3/2}} \exp(-\pi|v| + 2iv \log 2|v|/e - 3\pi i \operatorname{sgn}(v)/4) \cosh \pi v \quad (v \rightarrow \pm\infty), \quad (3.3)$$

which is of order $|v|^{-3/2}$ as $v \rightarrow \pm\infty$. The result follows as in Lemma 2.3.

(b) The proof of Theorem 1.1(b) is similar.

4. Maximal operators for Euclidean wave and Schrödinger operators

The Laplace operator $\Delta = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d; dx)$ and extends naturally to a closed linear operator in $L^p(\mathbb{R}^d; dx)$ for $1 \leq p < \infty$.

For this operator, d'Alembert's solution of the wave equation gives a representation of the cosine family as averages over the unit sphere with respect to normalized Lebesgue measure, namely

$$\cos t\sqrt{\Delta}f(x) = \int_{\{|y|=1\}} f(x+ty)\sigma_d(dy) \quad (f \in C_c^\infty(\mathbb{R}^d)). \quad (4.1)$$

Corollary 4.1. *Let $4d/(2d+1) < p < 4d/(2d-1)$. Then*

(a) *the maximal operator $W_* : \mathbb{D}_{L^p}(\log \Delta) \rightarrow L^p(\mathbb{R}^d)$ is bounded, where*

$$W_* : f \mapsto \sup_{t>0} \left| \frac{\sin t\sqrt{\Delta}}{t\sqrt{\Delta}} f(x) \right|; \quad (4.2)$$

(b) *similarly, the maximal operator $S_* : \mathbb{D}_{L^p}(\log \Delta) \rightarrow L^p(\mathbb{R}^d)$ is bounded, where*

$$S_* : f \mapsto \sup_{t>0} \left| \frac{\sin t\Delta}{t\Delta} f(x) \right|; \quad (4.3)$$

(c) *and for $1 > \gamma > \max\{(p+2pd-4d)/2p, (p+4d-2pd)/2p\}$, the maximal operator*

$$f \mapsto \sup_{t>0} \left| \frac{\cos t\sqrt{\Delta} - I}{(t\sqrt{\Delta})^\gamma} f(x) \right| \quad (4.4)$$

is bounded $L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$.

Proof. (a), (b) Using the Marcinkiewicz multiplier theorem and interpolation as in [5, Lemma 3.1], one can achieve the bound $\|\Delta^{iv}\|_{B(L^p(\mathbb{R}^d))} \leq C_{p,d}(1+|v|)^\beta$ with $\beta < 1/2$ for such p and d . Hence the hypotheses of Theorem 1.1 are satisfied.

(c) The Mellin transform of $G(x) = (\cos \sqrt{x} - 1)/x^{\gamma/2}$ is found to be $\tilde{G}(s) = 2\Gamma(2s - \gamma) \cos \pi(2s - \gamma)/2$, which is holomorphic on the strip $-1/2 < \Re s < 1/4$. This identity may be deduced from the Mellin transform formula for the sine function [20, p. 204] by integration by parts. Furthermore, by Stirling's formula and the bound on the growth of the group, the integral $\int_{-\infty}^{\infty} |\tilde{G}(iv)| \|\Delta^{-iv}\|_{B(L^p)} dv$ is convergent; and so we can argue as with (2.6). This may be compared to [5, Theorems 5.1, 5.2], and [9, Corollary 4].

Remarks 4.2. (a) With $d = 1$ and $L = \Delta$, it follows from (4.1) that $\sin t\sqrt{\Delta}/t\sqrt{\Delta}f(x) = \int_{-t}^t f(x+u)du/2t$ and the maximal operator W_* of (4.2) is simply the Hardy-Littlewood maximal operator [14, p. 237], so Corollary 4.1(a) may be improved to

the statement that W_* is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. However, Theorem 1.1 does not extend to the case of $p = 1$. For it is easy to check that $f(x) = \int_{-1}^1 e^{-(x-y)^2/2} dy / \sqrt{2\pi}$ is integrable and belongs to $\mathbb{D}(\log \Delta)$ (defined via the Fourier transform) in $L^1(\mathbb{R})$; but the maximal function of f is not integrable.

(b) The maximal operator $f \mapsto \sup_{t>0} |e^{it\Delta} f|$ is unbounded $L^p(\mathbb{R}) \rightarrow L^p_{loc}(\mathbb{R})$ for $1 \leq p < 2$ [12, Theorem 2].

The following results improve upon the cases $d = 1$ and $d \geq 3$ of Corollary 4.1(b) respectively. Possibly they are known, but I have not able to locate a reference.

Proposition 4.3. *Let Δ be the Laplacian over the real line. Then the maximal operator S_* for the sine family $\sin t\Delta/t\Delta$ defines a bounded sublinear operator $L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ for $1 < p < \infty$. Likewise, for the Laplacian over the circle, S_* is bounded $L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$ for $1 < p < \infty$.*

Proof. We can represent Δ by the Fourier multiplier $\Delta e^{i\xi x} = \xi^2 e^{i\xi x}$, so S_* may be viewed as the maximal convolution operator for the kernels $K_t(x-y)$ which represent the multipliers $\sin t\xi^2/t\xi^2$. Indeed, we have $K_t(x-y) = F((x-y)/\sqrt{t})/\sqrt{t}$, where by Fourier inversion

$$F(u) = \int_{-\infty}^{\infty} \frac{\sin \xi^2}{\xi^2} e^{i\xi u} \frac{d\xi}{2\pi} \quad (u \in \mathbb{R}). \quad (4.5)$$

By considering the Fresnel integral $F''(u)$, one can derive an asymptotic formula for F which implies $|F(u)| \leq C/(1+u^2)$ ($u \in \mathbb{R}$) for some $C < \infty$. It follows that the sine family satisfies

$$\begin{aligned} \frac{\sin t\Delta}{t\Delta} f(x) &= \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} F((x-y)/\sqrt{t}) f(y) dy \\ &\leq C \int_{-\infty}^{\infty} \frac{\sqrt{t}}{t + (x-y)^2} |f(y)| dy = CP_{\sqrt{t}} |f|(x), \end{aligned} \quad (4.6)$$

where $P_t = e^{-t\sqrt{\Delta}}$ is the Poisson semigroup for the upper half plane [14, p. 142]. By the Hardy-Littlewood maximal theorem [14, p. 237], the maximal operator $P_* : f \mapsto \sup_{t>0} |P_t f(x)|$ defines a bounded sublinear operator $L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ for $1 < p < \infty$; and hence by (4.6), S_* is likewise bounded on $L^p(\mathbb{R})$ for all such p .

To show that S_* is bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$, one uses the Poisson summation formula and the previous estimate to show that

$$\left| \sum_n \frac{\sin n^2 t}{n^2 t} e^{in(\theta-\phi)} \right| \leq \frac{C\sqrt{t}}{t + (\theta-\phi)^2} + C \quad (0 \leq \theta, \phi \leq 2\pi) \quad (4.7)$$

for $0 < t < 1$. The proof then proceeds as before.

Proposition 4.4. *For dimension $d \geq 3$ and $d/(d-1) < p \leq \infty$ there exists $C_{p,d} < \infty$ such that all $f \in C_c^\infty(\mathbb{R}^d)$ satisfy $\|S_*(f)\|_{L^p} \leq C_{p,d}\|f\|_{L^p}$.*

Proof. We can use the spectral theorem and the Fourier inversion formula with (4.5) to obtain

$$\frac{\sin t\Delta}{t\Delta}f(x) = \int_{-\infty}^{\infty} F(u) \cos u\sqrt{(t\Delta)}f(x)du \quad (4.8)$$

for smooth functions of compact support. Since F is integrable, the inequality $S_*f(x) \leq C \sup_{t>0} |\cos t\sqrt{\Delta}f(x)|$ follows by convexity. Stein [18, p. 518] has shown that the maximal spherical averaging operator $f \mapsto \sup_{t>0} |\cos t\sqrt{\Delta}f(x)|$ defines a strongly (p, p) bounded sublinear operator for this restricted range of p and d ; his proof depends upon Fourier transform restriction phenomena. See also [16].

5. Almost everywhere Cesaro summability of orthogonal series

Throughout this section we let μ be a probability measure on X , and let (φ_n) ($n \in \mathbb{N}$) be an orthonormal basis for $L^2(\mu)$. For $f \in L^2(\mu)$ we let $f \sim \sum_n a_n \varphi_n$ be the orthogonal series generated by f . For a real sequence (μ_n) ($n \in \mathbb{N}$) with $\lambda_n = |\mu_n| \geq 1$ and $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, we introduce a self-adjoint operator A in $L^2(\mu)$ by setting $A\varphi_n = \mu_n \varphi_n$ and extending linearly. Then $U_t = e^{itA}$ defines a C_0 unitary group on $L^2(\mu)$. We also introduce the positive self-adjoint operator L by $L\varphi_n = \lambda_n \varphi_n$, so L^{iu} is a C_0 unitary group on $L^2(\mu)$.

For $\delta > 0$, we say that $\sum_n a_n \varphi_n$ is Cesàro (C, δ) summable to f at x , if

$$\sum_{n:\lambda_n < N} \left(1 - \frac{\lambda_n}{N}\right)^\delta a_n \varphi_n(x) \rightarrow f(x) \quad (N \rightarrow \infty). \quad (5.1)$$

See [11]. Note that the each sum on the left-hand side is finite. We write $C_t^\delta f = (I - tL)_+^\delta f$ for the expression on the left-hand side, and $C_*^\delta f(x) = \sup_{t>0} |C_t^\delta f(x)|$ for the associated maximal function.

5.1 Proof of Theorem 1.3. We shall show that C_*^δ is a bounded sublinear operator $\mathbb{D}_{L^2}(\log L) \rightarrow L^2(\mu)$. Almost everywhere convergence of the (C, δ) Cesàro sums follows by standard and classical arguments. The Mellin transform of $G(x) = (1 - x)_+^\delta$ is

$$\tilde{G}(s) = \int_0^1 (1 - x)^\delta x^{s-1} dx = \frac{\Gamma(\delta + 1)\Gamma(s)}{\Gamma(\delta + 1 + s)}, \quad (5.2)$$

which defines a holomorphic function on the strip $R = \{s : -1 < \Re s < 1\}$, apart from a simple pole at $s = 0$. Further, Stirling's formula leads to the estimate $\tilde{G}(s) = O(|s|^{-1-\delta})$

uniformly for $s \in R$ as $|s| \rightarrow \infty$. Arguing as in the proof of Lemma 2.3, we deduce that the associated maximal operator $C_*^\delta f = \sup_{t>0} |G(tL)f|$ is bounded $\mathbb{D}_{L^2}(\log L) \rightarrow L^2(\mu)$.

Definition. A bounded linear operator U on $L^2(\mu)$ is said to be *separation-preserving* if, for any f and g in $L^2(\mu)$ with $fg = 0$ μ -almost everywhere, $(Uf)(Ug) = 0$ also holds μ -almost everywhere. A bounded linear operator U is said to be *positivity preserving*, if $Uf \geq 0$, whenever $f \geq 0$. Any C_0 -group U_t ($t \in \mathbb{R}$) of positivity preserving operators is also separation-preserving; see [1, p. 55].

Proposition 5.2. *Suppose that the unitary group $U_t = e^{itA}$ has U_t separation-preserving for each $t \in \mathbb{R}$. Then for any $f \in L^2(\mu)$ and $\delta > 0$, the series $\sum_n a_n \varphi_n(x)$ is (C, δ) summable to $f(x)$ for μ -almost all x .*

Proof. The proof is based upon a transference technique, where we pass from the translation group $\tau_u : f(x) \mapsto f(x+u)$ on the line to the unitary group U_t on $L^2(\mu)$. By the spectral theorem, we can write for any $f \in L^2(\mu)$

$$(I - tL)_+^\delta f(x) = \int_{-\infty}^{\infty} h_\delta(u) U_{tu} f(x) du / (2\pi) \quad (x \in X), \quad (5.3)$$

where h_δ is the Fourier transform of $(1 - |x|)_+^\delta$. One can show that $|h_\delta(u)| \leq \kappa_\delta (1 + u^2)^{-(1+\delta)/2}$ ($u \in \mathbb{R}$) for some $\kappa_\delta < \infty$. The operator $(I - tL)_+^\delta$ may thus be obtained by transferring to $L^2(\mu)$ the bounded convolution operator $\int_{-\infty}^{\infty} h_\delta(u) \tau_{tu} du$ on $L^2(\mathbb{R})$; further, by the theorem 2.3 of [1], the maximal operator for the family $(I - tL)_+^\delta$ may be obtained by transference of the corresponding maximal convolution operator. To check that the latter is strongly $(2, 2)$ bounded, we note that for any $g \in L^2(\mathbb{R})$ we have

$$\begin{aligned} \sup_{t>0} \left| \int_{-\infty}^{\infty} h_\delta(u) \tau_{tu} g(x) du \right| &\leq \sup_{t>0} \kappa_\delta \int_{-\infty}^{\infty} \frac{1}{(1 + u^2)^{(1+\delta)/2}} |g(x + tu)| du \\ &\leq \sup_{t>0} (1 + \delta) \kappa_\delta \int_0^{\infty} \frac{u^2}{(1 + u^2)^{(\delta+3)/2}} \frac{1}{u} \int_{-u}^u |g(x + tv)| dv du, \end{aligned} \quad (5.4)$$

where the last step follows from Fubini's theorem. The latest integral is a convex combination of averages of $|g|$, and so is strongly $(2, 2)$ bounded by the Hardy-Littlewood maximal averages theorem [14, p. 237]. By the transference theorem for maximal operators, $\sup_{t>0} (I - tL)_+^\delta$ is likewise strongly $(2, 2)$ bounded, as required.

This proof of Proposition 5.2 ultimately reduced to the following observation:

Proposition 5.3. *Suppose that $t \mapsto e^{itA}$ is a bounded and strongly continuous representation of \mathbb{R} on $L^p(\mu)$, for some $1 < p < \infty$, with each e^{itA} separation-preserving; and let $\delta > 0$. Then there exists $\kappa_\delta < \infty$ such that $C_*^\delta f \leq \kappa_\delta S_* f$ for all f ; and S_* is bounded $L^p(\mu) \rightarrow L^p(\mu)$.*

6. Subordination of maximal functions

Let X be a complete Riemannian manifold with metric ρ and measure μ . We let L be the Laplace operator in $L^2(\mu)$, which is positive and self-adjoint. The wave operator $\cos t\sqrt{L}$ has a distributional kernel $\cos t\sqrt{L}(x, y)$ supported on the strip $\{(x, y) \in X^2 : \rho(x, y) \leq t\}$ surrounding the diagonal $x = y$; thus the solutions to the wave equation are of finite propagation speed [8]. In view of Remark 4.2(a), it is natural to regard the maximal function W_* for the wave sine family $\sin t\sqrt{L}/t\sqrt{L}$ as the analogue of the Hardy-Littlewood maximal function.

The Poisson semigroup may be defined by

$$P_t f(x) = e^{-t\sqrt{L}} f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + u^2} \cos u\sqrt{L} f(x) du \quad (6.1)$$

for $f \in L^2(\mu)$, and it extends to define a symmetric diffusion semigroup on $L^p(\mu)$ for $1 \leq p \leq \infty$ [8, 17]. Here $V(t, x) = P_t f(x)$ defines a harmonic function, in the sense that $(\partial^2/\partial t^2 - L)V(t, x) = 0$ on $(0, \infty) \times X$. The maximal functions for the Poisson and heat semigroups are denoted $P_* f(x) = \sup_{t>0} |P_t f(x)|$ and $H_* f(x) = \sup_{t>0} |H_t f(x)|$ respectively. These may be compared pointwise in the following result.

Proposition 6.1. *The maximal functions satisfy:*

- (a) $P_* f(x) \leq H_* f(x) \leq W_* f(x)$ for all $f \in C_c^\infty(X)$ and $x \in X$; and
- (b) $C_*^\delta f(x) \leq \kappa_\delta S_* f(x)$ for some constant κ_δ where $1 < \delta < 2$.
- (c) $S_* f(x) \leq \kappa P_* f(x)$, whenever $\cos t\sqrt{L}$ ($t > 0$) is positivity preserving.

Proof. The Poisson and heat semigroups $H_t = e^{-tL}$ are related by the Bochner subordination formula [17, p. 47]

$$P_t f(x) = \int_0^\infty H_{t^2/v} f(x) e^{-v} dv / \sqrt{v\pi}; \quad (6.2)$$

whereas one can obtain the relation

$$H_t f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty s^2 e^{-s^2/2} \frac{\sin s\sqrt{(2tL)}}{s\sqrt{(2tL)}} f(x) ds \quad (6.3)$$

by integration by parts and Fourier inversion [8]. And so by convexity the maximal functions satisfy (a).

(b) We take $1 < \delta < 2$ and introduce $h_\delta(u) = \int_{-1}^1 (1 - |x|)^\delta \cos uxdx$, which is a continuously differentiable and integrable function on \mathbb{R} . Using repeated partial integration, one shows that $\sigma_\delta(u) = -uh'_\delta(u)$ is integrable over \mathbb{R} . By Fourier inversion and integration by parts, one obtains

$$(1 - tL)_+^\delta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin utL}{utL} \sigma_\delta(u) du \quad (6.4)$$

from the spectral theorem. Now (b) follows by convexity, with $\kappa_\delta = \int_{-\infty}^{\infty} |\sigma_\delta(u)| du / 2\pi$.

(c) This follows from (4.8) and (6.1) since $F(u) \leq C/(1 + u^2)$.

Remarks 6.2. (1) On account of (4.1), the hypothesis of 6.1(b) is satisfied by the Euclidean Laplace operator.

(2) The inequality 6.1(b) is the basis for Hardy's theorem on the almost everywhere summability of classical Fourier series [11, Theorem 3] over \mathbb{T} ; the method of Kolmogorov-Seliverstov is rather different in that it involves detailed analysis of kernel functions [13].

(3) Results concerning spherical sums of classical Fourier series over \mathbb{T}^d with $p \neq 2$ are sensitive to dimension. Carleson and Sjölin showed that the operators $(I - t\Delta)_+^\delta$ are bounded $L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$ for $4/3 \leq p \leq 4$ for $t, \delta > 0$. Using our method with [5, Lemma 3.1] and (5.2), one obtains the weaker result that the $(I - t\Delta)_+^\delta$ ($t > 0$) are uniformly bounded $L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$ for $4/(2 + \delta) < p < 4/(2 - \delta)$. By contrast, $\delta = 0$ gives the multiplier for the disc, which is famously unbounded on $L^p(\mathbb{R}^2)$ for $p \neq 2$ [10].

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